THE

MATHEMATICAL GAZETTE.

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THE TEACHING OF MATHEMATICS.

WHILE all teachers of Mathematics should be interested in the paper read last September by Prof. Perry and in the debate which followed it, members of our own Association, founded as it was for the Improvement of Geometrical Teaching, should give it special attention and should take advantage of the upheaval at Glasgow to press some of their immediate aims. Although Prof. Perry presented his case with a certain amount of picturesque exaggeration, I feel sure that his contentions were in the main true, and that the mathematical training of the schools, however well fitted for those who are being prepared for a University course, turns out the ordinary boy with a modicum of Mathematics which he finds of little use for any professional purpose. I do not however think that we need look upon Prof. Perry as an opponent but rather as a valuable ally. I am convinced that much might be done, at any rate in the earlier stages of mathematical teaching, which besides going far to meet Prof. Perry's views and to remove the reproach (in my opinion deserved) of the Engineer, would be a considerable gain from our own point of view. Our cause may be advanced considerably if we make a proper use of the present opportunity.

The debate at Glasgow ranged widely and there was some vagueness of aim on the part of some of the speakers; but three points emerged with clearness on which there seemed a fair amount of agreement. I will take them under the heads of (i) Examinations (ii) Text-books (iii) Intuitional or Experimental Geometry.

(i) It was admitted that the school system is dominated by examinations, and that little could be done unless Examiners and Inspectors of schools modified the nature of their requirements. A lamentable case of the blighting influence of reactionary examiners was brought forward by Prof. Silvanus Thompson.

(ii) The question of the retention of Euclid as a text-book was again raised. It was urged with considerable force that our retention of a book discarded by other nations had at any rate a presumption against it, and that it was wrong to sacrifice the interests of education to the ease of the examiner.

(iii) More than one speaker pointed out that if the experimental or intuitional method of introducing the truths of Mathematics, and especially of Geometry, were used from the lowest classes of our schools upwards, the strictly deductive course would not lose but gain in effectiveness. I used the greater part of the ten minutes allowed me to urge this point,

and need not enlarge upon it here, though it seems the one thing in which we can all of us work *individually* for improvement without waiting for corporate action. Prof. Perry, in the report of the debate he is about to publish, has kindly allowed me to add some further considerations in favour of the course advocated.

Now the Association seems bound to have something to say in its corporate capacity on each of subjects (i), (ii), and (iii).

As to (i) it might appoint a small committee to keep itself in touch with that appointed by the British Association, and to give the views of the latter, when approved, all the support which, as an organised body of mathematical teachers, it can give, in bringing useful pressure to bear on Examiners and Inspectors.

(ii) It might, on its own account, again approach the Universities and other examining bodies on the subject of text-books, urging, as a sound reason for re-opening the question, the opinion so strongly expressed at Glasgow of the unfitness of Euclid as a text-book for beginners, and asking as first instalment of a much-needed reform, the official recognition of its own work, as a text-book for Examinations. I have always thought that the Council of the A.I.G.T. lost, in 1887, a good opportunity through its liberality of sentiment with regard to other text-books. If it had merely asked that its own should be officially recognised, I fancy it might have succeeded, at any rate at Cambridge; and when no great harm had been found to follow the first step, the authorities would have allowed, by degrees, the greater freedom required. As an old Hon. Sec. it was with great regret that I heard one speaker talk as if the A.I.G.T. had never taken the trouble to bring out a text-book of its own. Such a work not only exists, but has been used with marked success in the Colonies. It was mainly the work of two experienced teachers—Mr. R. B. Hayward of Harrow and Mr. R. Levett of Birmingham—but was circulated in proof among the members of the A.I.G.T. for remark, and only published after minute and searching criticism. It should have received some better recognition than that accorded to it by Oxford and Cambridge, and should have been admitted as a successful attempt—possibly to be followed by other and still more successful attempts—to get an improved text-book of the elements of geometry.

(iii) The Association should formally pass a resolution insisting on the introduction of an intuitional or experimental course of Geometry into all schools to prepare for and accompany the present rigidly deductive course.

I venture to hope that some official action will be taken on these points, and that some of the younger members of the Association with whom the future lies will throw themselves heartily into the work.

EDWARD M. LANGLEY.

THE TEACHING OF MATHEMATICS—A COMPROMISE.

AFTER Professor Perry's stimulating denunciation at Glasgow, many of us must be wondering how far are we really able to mend our ways at public schools? What about Army, Navy, and University examinations? We wish to produce useful engineers, but we must not be too independent; examination requirements are always urgent. Now it may be admitted that the elementary tests imposed by the Universities are quite hopeless. On the other hand, both Navy and Army examiners seem to have been converted by Professor Perry.

Again, how are we to fit the new teaching on to the knowledge which the boy brings to his public school? And what will be the practical details of the arrangement?

I venture to trespass upon your space with a scheme of work which has stood the test of some little experience. We have used it here for the last eighteen months, and are not, I think, dissatisfied with it. It may be described as a compromise; but we hope that Professor Perry, in an indulgent mood, would not condemn it utterly.

SCHEME OF MATHEMATICAL TEACHING.

TERM.	GEOMETRY.	ARITHMETIC.	ALGEBRA.
I.	Experimental work with ruler, compass, protractor, and set squares; paper cutting.	Prime factors, leading to H.C.F. and L.C.M.; fractions; drill in decimal notation; addition and subtraction of decimals; multiplication and division of decimals by single figure; applications of fractions and decimals to concrete quantity; unitary method.	Transition from Arithmetic to Algebra; negative number and its application to quantity; numerical evaluation; use of brackets; drill in notation of Algebra; addition, subtraction, and multiplication, regarded as operations with brackets; positive integral indices; simplest equations and problems.
II.	Euclid I. 1, to I. 26 (omitting 2, 3, 7). Definitions as required. Throughout the Geometry course plenty of easy riders. Geometrical drawing and exercises in measurement to go on side by side with theory.	Multiplication of decimals beginning with left-hand figure of multiplier; division of decimals; decimalisation of money; how recurring decimals occur when we turn a fraction into a decimal; metric system; revision of Term I.	Simple numerical equations in one variable, to be solved in all cases by the use of the four axioms, solutions to be verified always; symbolical expression; problems; easy factors; revision of Term I.
III.	Euclid I. 27—I. 48; part of Book III.	Areas of rectangles, parallelograms, triangles, trapezia, and irregular figures, chiefly by means of squared paper; square and cube root by prime factors; graphical treatment of square root; volumes; simple interest.	Cartesian coordinates; plotting simple graphs; plotting tables of statistics; simultaneous simple equations illustrated by graphs (literal equations to be postponed); problems on simultaneous equations; easy factors and fractions.
IV.	Euclid III., with the exception of Propositions 35, 36, 37.	Rule for square root; how to deal with $\frac{1}{\sqrt{2}-1}$, etc.; contracted multiplication of decimals, with applications (practice; simple and compound interest; involution, etc.).	Factors, leading to H.C.F. and L.C.M.; long division; detached coefficients; method for H.C.F. when factors are not obvious; harder fractions; manipulation of quadratic surds; graphs.
V.	Euclid II., and Propositions 35, 36, 37 of Book III.	Contracted division of decimals, with applications (inverse interest, present worth, discount, etc.); ratio and percentage.	Connections between factors and equations; quadratic equations with problems; reducing degree of equation when one root is obvious.
VI.	Euclid IV., Propositions 1-5, 10, 11, 15. The whole book to be treated as a collection of problems in geometrical drawing.	Revision of contracted multiplication and division; stocks and shares.	Remainder theorem; arithmetic and geometric progressions, without formulae; ratio and proportion.
VII.	Euclid VI., omitting Euclid's treatment of proportion, and substituting an algebraic treatment.	Miscellaneous problems.	Indices, so far as needed to introduce common logarithms; arithmetical problems involving use of logarithm tables; variation; revision of progressions, introducing formulae for same; recurring decimals.
VIII	Revision of Books III. and VI.	Numerical trigonometry of the acute angle, using sine, cosine, tangent; problems of heights and distances; solution of right-angled triangles, at first by natural sines, etc., afterwards by logarithmic sines, etc.; solution of general triangles by drawing perpendiculars and thus dividing into right-angled triangles. Irrational equations; simple cases of simultaneous quadratics, especially such as can be illustrated by graphs; literal equations; easy permutations and combinations.	

After this stage the following work is taken:
Geometry of the triangle, solid geometry, surfaces and volumes of parallelopipeds, tetrahedra, cylinder, cone, sphere; modern geometry (similar figures, harmonic ranges and pencils, inversion, coaxal circles, pole and polar, cross ratio and involution).
Elementary Trigonometry.
Elementary Dynamics.
Binomial theorem for positive integral index; partial fractions; limits and convergency; general binomial and exponential theorems.
Vectors and complex quantity.
Graphs; asymptotes; slope of a curve; area of a curve.

It should be explained that the amount of time available for this work is, roughly, four hours a week, together with preparation. Many boys would repeat a term's work; but the really bright boy is able to take the work as it is set down, term by term.

CHARLES GODFREY.

THE COLLEGE, WINCHESTER,
Oct., 1901.

FROM A PUBLIC SCHOOL POINT OF VIEW.

THE course of Mathematics suggested by Professor Perry at the British Association "is recommended for training colleges and for boys and girls." It may be admirably adapted to the wants of training colleges, but seems quite impracticable for public schools, though, as we shall see later, there are several suggestions that might well be adopted.

Arithmetic.—"Decimals to be used from the beginning; the fallacy of retaining more figures than are justifiable in calculations involving numbers which represent observed or measured quantities. Contracted and approximate methods of multiplying and dividing numbers whereby all unnecessary figures may be omitted." This is all admirable in principle, but in practice it can be pushed too far; in using approximate methods too soon there is danger of teaching the boy to undervalue accuracy. He says, "this answer is only twopence out," but does not consider the difference between being twopence out in a shilling and twopence out in £100; and again, he does not consider that perhaps his method is wrong, and that this causes the inaccuracy. To take a rather exaggerated example: in finding the true discount on £100 due in a year at 2½ per cent., a boy perhaps finds the interest on £100 for a year, and says that it is less than fifteen pence out. It may be answered that true discount is seldom used, but if this mistake is not corrected, a wrong impression is made on the boy's mind. Professor Perry lays special stress on calculating percentages: surely the first thing is to learn on what the percentage is to be taken. Far and away the most important thing in using approximate methods is to know to what degree of accuracy we are working. This is very difficult for young boys, and, in general, it is better at first to give them carefully chosen examples in which there is no need for approximate work. This enables them to concentrate their thoughts on the methods they are using and the meanings of the technical terms that occur (*e.g.* interest, discount, quotation, etc., etc.). Then, when they have grasped these, by all means apply approximate methods, and point out that "it is dishonest to use more figures than we are sure of."

"The use of the slide rule" can be taught in a very short time to a boy who has had a fairly good school training; it might perhaps be put in the hands of the older boys, but it would be extremely unwise to give it to a young boy.

"Using algebraical formulae in working examples on ratio and variation." This suggestion is so startling that one could hardly believe it. On referring, however, to Professor Perry's *Practical Mathematics* we find the following: "Compound Proportion.—If 10 labourers dig 150 yards of trench in 5 days of 12 hours, how many labourers will dig 356 yards in 2 days of 9 hours? If

we use l, y, d, and h, we see that the assumption made is

$$l \propto \frac{y}{dh}, \quad \text{or} \quad l = a\frac{y}{dh}.$$

Hence $$10 = a\frac{150}{5\times 12}, \text{ so that } a = \frac{5\times 12\times 10}{150} = 4.$$

So that we have the formula $l = 4\frac{y}{dh}$ for working any exercise. Thus the answer wanted is

$$l = 4\frac{356}{2\times 9} = 79{\cdot}11 \text{ labourers."}$$

This sort of method shows entire ignorance of the state of the youthful mind. A boy never goes from the general to the particular; in his mind the concrete must always precede the abstract. A young boy learning how to tackle an example involving letters, would first be told to take a *numerical* value for each *letter* in turn, and to consider what he would do with the numbers. He would then be led on to deal in the same way with the letters; in fact, Professor Perry's order would be inverted. What would he say if we followed Professor Cayley, and taught, first of all, geometry of n dimensions, and then descended to the mere particular cases when $n=2$ or 3? This would be quite analogous to the principle proposed above.

There is no mention of the *unitary method*. Judging from the above example in compound proportion, the omission is intentional. There is no better training than the solution of problems by unitary method; its application develops the power of careful thought (as opposed to mere effort of memory) more than anything else in arithmetic. All questions on percentages, interest, present worth, discount, stocks, etc., are best solved by applying this method to the definitions. Professor Perry would treat all these "as the simplest of algebraic exercises."

Algebra.—"Being told in words how to deal arithmetically with a quantity, to be able to state the matter algebraically." This is the method adopted by every teacher in his first lessons in algebra, but surely it is a direct contradiction of the principles involved in the method for compound proportion quoted above. But why are the "Rules of Indices" put before this natural bridge between arithmetic and algebra"? This part of the course is very indefinite. It is difficult to understand exactly what work is included in "Practice in algebraic manipulation generally."

Geometry.—In introducing his scheme Professor Perry remarked, "I think that men who teach demonstrative geometry and orthodox mathematics generally are not only destroying what power to think already exists, but are producing a dislike, a hatred for all kinds of computation, and therefore for all scientific study of nature, and are doing incalculable harm." Yet in the actual scheme he says, "A good teacher will occasionally introduce demonstrative proof as well as measurement," and in the advanced course he suggests "Demonstrative geometry based upon Euclid." This is rather contradictory; perhaps we may conclude from the last quotation that he wishes the subject to be still taught, but wants Euclid to go. The result would be chaos; there would immediately be ten thousand text-books treating the subject in ten thousand different orders (compare the text-books on geometrical conics). Of course they have abandoned Euclid on the Continent and in America, but the circumstances are rather different. In England between 12 and 14 a boy goes from a preparatory school to a public school. This would probably necessitate a change of text-book and consequently a change of order of propositions. For some time he would not understand the master and the master would not understand him. This is not the case in foreign countries, where the schools are largely under State control (the English public schools rejoice—rightly or wrongly—in thinking this a long way off

for them), and so there is more uniformity: and in Germany, for instance, the boy has the same teacher throughout his nine years' course in the Gymnasium, Realgymnasium or Oberrealschule (which to some extent correspond to our Public Schools), and his examination, at the end of his course there, is not for boys of different schools, but is limited to one, so that it does not matter what text-book is used so long as its propositions follow one another in a logical sequence.

Considering the complexity of our educational system in England, it seems to be essential that we should have a standard order for propositions in demonstrative geometry. The only possible chance of any new order being agreed to is its adoption by the great examining bodies, such as the Universities and the Civil Service Commissioners, and the most hopeful of reformers would never hope for that. Why not then be contented with Euclid's order (*i.e.* in proving any proposition in Euclid no proposition may be assumed that does not precede it*)? Surely this allows enough latitude for reform. The proofs of many propositions might be omitted in a first reading, and some might be omitted altogether; the Third Book (except 35-37) might be taken before the Second; the Eleventh Book (except Prop. 17) might be taken at any time after the first thirty-three propositions of the First Book.

The most marked feature of Professor Perry's scheme is the large amount of practical work he suggests under the headings of *Geometry*, *Mensuration*, and *Use of Squared Paper*.

Some such course is adopted in all continental countries. In America they have a very long one,† and some schools in England are doing work of that character, but it is not general, and there is not enough of it in any of the preparatory and public schools. It would be well for every boy to go through such a course with his mathematical master (the science masters do something of the kind in practical physics, but it cannot, in this case, be interwoven with the mathematics, and so loses much of its value). The majority of the work proposed can be done in an ordinary classroom, each boy being provided with a good set of mathematical instruments (including ruler, compasses, set squares, and protractor) and a pair of scissors. Work in which weighing is necessary must be done in a laboratory (it might perhaps be found necessary for this to be done by the science masters, but in any case it should run hand in hand with the mathematical teaching).

Mathematical masters must feel some diffidence in criticising the details of the work proposed, for though many of them have done practical work in physics, yet as a rule they have not much first-hand experience of how this work is done by boys; nevertheless one may venture to note two points in the scheme. (1) The *Mensuration* work consists very largely of *verification of rules* for areas and volumes; it may be a mere matter of language, but is it not preferable, if we must have rules, that the work should be *invention* of rules? (2) Considering the extraordinary slowness with which boys do practical work of this type, the amount proposed seems rather large for an elementary course (a young boy could hardly be expected to use a planimeter with advantage or bother his head with formulae for the volume and surface of a ring). Possibly the pace would increase when a really good course of simple measurement was started before demonstrative geometry (instead of after, as is too often the case at present).

I need hardly point out that Professor Perry seems to have arranged his scheme purely for the sake of enabling the boy to attain a certain mechanical

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* See reports of the Cambridge Mathematical Board and the Oxford Board of Natural Science.

† I have not been able, in the limited time I have had for writing this article, to ascertain when this course begins—certainly demonstrative geometry is not begun till the age of 16 or 17. Further information can be obtained from D. E. Smith's *Teaching of Elementary Mathematics*.

power (which, being mechanical, would soon rust unless constantly used). No thought is given to the development of brain power; in fact such a scheme would deaden all power of thinking. The man it would produce might have a fair power of applying a formula, but would not be able to invent one, and any problem that was not of the stereotyped kind he could never attack with the slightest hope of success.

So far we have chiefly looked at the faults of Professor Perry's scheme. Let us briefly consider a few of his suggestions that could be adopted with advantage.

"The use of rough checks in arithmetical work, especially with regard to the position of the decimal point" might be more generally applied than at present. Too much stress cannot be laid on the value of the use of squared paper for mapping statistics, finding areas, solving equations, etc., etc. There is no doubt that its cost has, in the past, prevented it being used in teaching mathematics, but now that it is quite cheap it should be used freely. The practical geometry course should commence sometime before Euclid is begun, and continue side by side with it. The early introduction of trigonometrical ratios and problems in heights and distances would, with squared paper and experimental geometry, add a very real interest to our mathematical teaching.

No attempt has been made in the above to draw up a scheme for this new work (if we may call it so). There are many practical details to be considered.* It would be of great value if masters would, through the *Mathematical Gazette*, let other masters know the result of any experiments they make in improving the teaching of elementary mathematics.

Professor Perry goes on to give an advanced course. This I have left alone, though it is quite possible to modify the work in our middle and lower forms. Examinations compel us at present to teach on somewhat orthodox lines in higher forms, but if the reform is started where feasible, may it not in time become possible even in higher forms to make our methods more modern?

One noteworthy remark Professor Perry makes:—"A good teacher must understand that no examination made by anyone other than himself can be framed which will properly test the result of his teaching." This principle is already observed in the high schools of Germany.

A. W. SIDDONS.

HARROW.

MATHEMATICAL NOTE.

103. [D. 6. b.] *Note on the Logarithmic Series.*

From the exponential series, if u is positive,

$$e^u > 1+u,$$

hence

$$u > \log(1+u),$$

and

$$\log\frac{1}{1-v} < \frac{v}{1-v},$$

where $u=\frac{v}{1-v}$, so that v is positive and <1.

Again, $e^v < \frac{1}{1-v}$, from the series, so that $\log\frac{1}{1-v} > v$.

Let $1-v=(1-x)^n$, so that $0<x<1$, and $v=1-(1-x)^{\frac{1}{n}}$.

*Mr. Hurst has written to *Nature* (Feb. 14th 1901) and Mr. Eggar to the *School World* (Oct. 1901) on the subject.

Thus $$\frac{1}{n}\log\frac{1}{1-x}<\frac{1-(1-x)^{\frac{1}{n}}}{(1-x)^{\frac{1}{n}}},\text{ but }>1-(1-x)^{\frac{1}{n}},$$

or $$\log\frac{1}{1-x}<n\left[(1-x)^{-\frac{1}{n}}-1\right],\text{ but }>n\left[1-(1-x)^{\frac{1}{n}}\right].$$

Now, by the Binomial Theorem,

$$n\left[(1-x)^{-\frac{1}{n}}-1\right]=x+\frac{1+\frac{1}{n}}{2}x^2+\frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{2.3}x^3+\ldots=A,\text{ say};$$

$$n\left[1-(1-x)^{\frac{1}{n}}\right]=x+\frac{1-\frac{1}{n}}{2}x^2+\frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{2.3}x^3+\ldots=B,\text{ say}.$$

Let $$C=x+\tfrac{1}{2}x^2+\tfrac{1}{3}x^3+\ldots.$$

Then $$A>C>B,$$

and $$A>\log\frac{1}{1-x}>B.$$

But $$A/B=(1-x)^{-\frac{1}{n}},$$

which can be made as near to 1 as we please by taking n great enough. Hence it must be true that

$$\log\frac{1}{1-x}=C=x+\tfrac{1}{2}x^2+\tfrac{1}{3}x^3+\ldots.$$

This holds when $0<x<1$. Writing x^2 for x we have

$$\log\frac{1}{1-x^2}=x^2+\tfrac{1}{2}x^4+\ldots.$$

Hence, by subtraction,

$$\log(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\ldots,$$

which therefore holds when $-1<x<1$.*

Again, write $\frac{1}{n}$ for v in the above. Thus

$$\log\frac{n}{n-1}<\frac{1}{n-1}\text{ and }>\frac{1}{n}.$$

Let S_n denote the sum to n terms of the series

$$1+\left(\frac{1}{2}-\log 2\right)+\left(\frac{1}{3}-\log\frac{3}{2}\right)+\ldots+\left(\frac{1}{n}-\log\frac{n}{n-1}\right)+\ldots,$$

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*The above proof of the logarithmic series holds only when x is real. We see however that when x is positive the sum of the series $A-C$, which consists of positive terms, tends to the limit zero as n increases. This will therefore also hold when x is complex, and thus in any case when $|x|<1$,

$$C=\lim_{n=\infty} n\left[(1-x)^{-\frac{1}{n}}-1\right]$$

$$=\text{a value of }\log\frac{1}{1-x},$$

but which value is to be taken has to be decided by considerations of continuity.

which is absolutely convergent, since its n^{th} term is numerically less than $\frac{1}{n-1}-\frac{1}{n}$. Then

$$S_n=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n$$

$$=2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right)-\log n,$$

$$S_{2n}=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2n}-\log 2n.$$

By subtraction,

$$S_{2n}-S_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}-\log 2.$$

But $S_{2n}-S_n$ diminishes without limit as n increases, being in fact numerically $<\frac{1}{2n}$, and therefore

$$\log 2=\operatorname*{Lim}_{n=\infty}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{2n}\right).$$

Similarly $\log 3=1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{1}{5}-\frac{2}{6}+\frac{1}{7}+\frac{1}{8}-\frac{2}{9}+\ldots,$

and so on.

A. C. Dixon.

REVIEWS AND NOTICES.

Die partiellen Differential-Gleichungen der mathematischen Physik. Nach Riemann's Vorlesungen in vierter Auflage neu bearbeitet von Heinrich Weber. Zweiter Band. Pp. xii., 528 (Braunschweig: Vieweg u. Sohn, 1901).

After an introductory section devoted to the hypergeometric series, Riemann's P-function, and the ordinary linear differential equation of the second order, Professor Weber deals in succession with the conduction of heat, elasticity, electric vibrations, and hydrodynamics. The English reader will here find himself, for the most part, on familiar ground, and may be excused if he reflects with some satisfaction that in this region at any rate his countrymen have done their full share both in discovery and in exposition. The general plan of the work (now complete) has been carried out with consistency and success, and the result cannot fail to prove very serviceable, not as a treatise on mathematical physics (which it does not profess to be), but rather as a guide to the spirit and methods of those parts of modern analysis which are most important in dealing with physical problems. By an attentive study of this treatise the student of physics will obtain a working knowledge of the Fourier analysis, spherical harmonics, Bessel functions, vector calculus, and the theory of the potential; all with direct reference to definite physical problems. Besides this, the variety of subjects dealt with tends to bring out the analogy and correspondence between different branches of applied mathematics which so often proves helpfully suggestive. Finally, it is in a certain sense advantageous to have the fundamental facts and assumptions of mathematical physics laid down in a clear-cut analytical form: this enables us to realise precisely the value of our analysis as an interpretation of Nature, and rightly used, is of service to the experimentalist, by suggesting definite lines of research, and by saving him misdirected and unprofitable labour.

The sections of this volume which will probably be found most interesting by experts are those on electric vibrations and on the motion of a gas. In the first of these Professor Weber discusses Maxwell's equations, and investigates in considerable detail the propagation of current along a wire and the electric oscillations of a spherical conductor surrounded by a dielectric. In the section on the motion of a gas we have an exposition of Riemann's theory of discontinuous motion, which, it will be remembered, has been criticised by Lord Rayleigh as being inconsistent with the principle of energy. Professor Weber's answer to

the objection is that Riemann's analysis leads to a result which, in its general form, implies the existence of impulsive waves of condensation *and* rarefaction; it is only waves of the latter kind which violate the principle of energy, so that Riemann's investigation appears to be unobjectionable so far as impulsive condensation is concerned. The case is somewhat analogous to problems in potential, etc., where a differential equation supplies us with two independent functions, one of which has to be rejected from physical considerations. G. B. MATHEWS.

The Proceedings of the Edinburgh Mathematical Society (Session 1900-1901). Vol. XIX. contains fifteen papers by various members of the Society. The most interesting is a note by Mr. David Mair, who proves in a very simple way that the nth root of a prime number cannot be the root of an equation of degree less than n with rational coefficients. Attention may be called to Mr. Carslaw's paper on the oblique incidence of a train of plane waves on a semi-infinite plane. G. B. MATHEWS.

The Teaching of Mathematics in the Higher Schools of Prussia. By J. W. A. YOUNG. Pp. xiv., 141. 2s. 6d. net. 1901. (Longmans.)

The title of this little book should ensure it a good circulation among our Association, founded as it was for the improvement of geometrical teaching, and enlarged to the wider scope indicated by its present name. But while mathematical teaching is the principal object of Dr. Young's investigations, it is by no means the only one. The general status of the higher schools, their relation to the Government, the position and emoluments of teachers, and their influence on mathematical as on other teaching, examinations, are all dealt with thoroughly. It should be studied carefully by all who profess to have any opinions on education, and especially by the governors of our public schools.

Those familiar only with the traditional nature of the appointments in English schools would find matter for reflection in the fact that it has been found possible to secure marked and peculiar success by methods entirely different. They will not, perhaps, be surprised to find that in a nation so distinguished for scientific discipline as the Germans, it is considered necessary for a man to prepare himself seriously for his life's work. Indeed there have been those in England, and some head masters among them, who have thought that preparation and training were as necessary to a schoolmaster as to a dentist. But it will possibly come as a surprise to many to find that in Germany teachers in secondary schools are valued civil servants; that while their salaries are moderate they are relatively considerable, being on the same scale as those for judges; that marked deference is paid to age and experience, responsible posts being usually allotted to elderly men (the market value of a German teacher increasing as he gets older!); that security of tenure seems to produce none of the bad effects that have been predicted for it; and that there is a State pension for every teacher on his retirement. The last two considerations may well make teachers in English schools envy the position of their German brethren, who can work steadily in their profession and give a life's devotion to it, secure that they will not be turned off at forty to get their living by tomato-growing. Turning to the more immediate subject of investigation, full particulars as to the hours allotted to Mathematics in various kinds of schools, and in their various classes, are supplied in a tabular form. The specimens of final examination papers given seem to be of a searching character, and suited, not only for testing a candidate's acquaintance with theory, but his power of applying theory to numerical calculation. Some model lessons are also given; they are well worth the study of those who have to teach the elements of any mathematical subject, but do not appear to differ from those which painstaking teachers in this country, who have taken the trouble to learn their craft, have been in the habit of giving. Few teachers, we hope, nowadays, think that ordinary boys can learn Mathematics by reading text-books or by the mere setting and marking of examples. On the other hand we share the doubts, which Dr. Young seems to have, as to whether the opposite plan of depending simply on oral instruction is not carried to too great lengths in German schools, and we are inclined to think that a boy should, by the age of 14 or 15, be getting into the habit of reading a text-book intelligently, and studying on his own account. In conclusion, we wish the book the wide circulation which its subject deserves and its price renders possible. E. M. LANGLEY.

Geometrical Exercises from Nixon's "Euclid Revised," with Solutions. By A. LARMOR, M.A. Pp. vi., 170. 3s. 6d. 1901. (Clarendon Press.)

From its first appearance Nixon's *Euclid Revised* has held a place of honour on the mathematical bookshelf. Not only did it present certain very novel and striking features which were entirely due to its talented author, but it embodied to a large extent the revolutionary ideas of the A.I.G.T.; it discarded the "Pons Asinorum," suppressed useless propositions, and generally shed both sweetness and light upon the dust-encrusted text. To those who, more or less unwisely, neglected to look at their Euclid until the night before an examination it proved a veritable boon, for by its aid the foolish virgin could run through the whole subject in a few hours. The presentation of the various proofs in a kind of "concentrated tabloid" form proved most attractive. Its various Addenda contained full demonstrations of those standard theorems which are intimately connected with the text—then only to be found in M'Dowell's *Exercises* and in Casey's *Sequel*. In addition to this, about nine hundred unsolved riders were proposed, forming on the whole a very complete geometrical armoury.

The present volume contains what we may be permitted to call "miniature" solutions of these riders. We learn from the preface that they were undertaken at the request of Mr. Nixon, who placed at the disposal of the author a large collection of material which he himself had accumulated towards this object before he was overtaken by ill-health. The *raison d'être* of the book is very well set forth:—"In Geometry more independent initiative is demanded, while the directness of the argument and the compactness and elegance of the solution provide a vigorous mental stimulus."

Of the compactness and elegance of the solutions here set forth there can be no possible manner of doubt. Perhaps in some cases the beginner might justifiably ask for a little more light, although in the latter portions the proofs are relatively fuller. Enunciations are given in large type, so that the volume is complete in itself. But in all cases the reader *est prié de faire la figure*. After the beginner has proceeded some little way in Euclid's text and is confronted with some easy theorem of this sort (say)—"the joins of the mid points of the sides of a quadrilateral form a parallelogram"—he has to exercise for the first time a certain intuition as to the shortest and most effective plan of campaign. Later, after much practice and after many failures, this power of intuition becomes much strengthened, and the student finally attains what Jacobi called "glückliche Divination." But, where and when the path he is treading seems to lead over difficult or impassable ground, he must have recourse to the services of a trusty guide to assist, warn, and instruct him on his way. He must grasp the fact that the work done by Euclid two hundred years B.C. is to-day as rigid and useful as ever it was; and also that if on the one hand there is no "royal road" in Geometry, on the other there are no steps to be retraced. One important stimulus in Geometry is the discovery of the value of transformations. Take, for instance, the Similitude theorem:—"Given two triangles whose sides are parallel, then the joins of corresponding points are collinear"—which is perfectly clear from the Sixth Book. Reciprocate it, and it leads to the fundamental theorem of Perspective. Project it, and it leads to the conception of parallel straight lines meeting on the line at infinity.

But to proceed to details.

No. 22, p. 23, and No. 30, p. 24.—If AXB be a right angle, and X either the mid point of AB or the foot of the perpendicular from X on AB, we have in both cases the relation $AP.PB - XP^2$. It therefore follows that if $ABCD$ be a cyclic quadrilateral whose diagonals intersect in X, then the mid points of the sides and the feet of the perpendiculars from X on the sides are eight points all lying on the circular locus $OP^2 + XP^2 = OA^2$, where O is the centre of the circle $ABCD$. Also the quadrilateral formed by joining these feet is noticeable as possibly the simplest instance of a quadrilateral circumscribed to one circle and inscribed to another.

No. 54, p. 69.—It is unnecessary to employ three cyclic quadrilaterals. By construction $ABXQ$ is cyclic; hence $B\hat{X}P = A = B\hat{C}P$, and $BCXP$ is also cyclic.

No. 52, p. 152.—Another method of constructing a triangle of given species with its vertices on three given concentric circles may be noticed. Let ABC be any triangle of the given species; r_1, r_2, r_3, the radii of the circles centre O. Describe a triangle $O\alpha P$, having its sides $O\alpha = r_1$, $\alpha P = (b/a)r_2$, $PO = (c/a)r_3$. Upon either

side of OP describe a triangle $PO\gamma$, similar to ABC, and upon $\alpha\gamma$ describe a triangle $\alpha\beta\gamma$ directly similar to $PO\gamma$. Then $\alpha\beta\gamma$ is the triangle required.

In the solution of No. 74, p. 11, it is shown that the theorem that "bisectors of the internal angles of a quadrilateral form a quadrilateral whose opposite angles are supplementary" is not always true. Beginners are always apt to draw their angles acute-angled and their quadrilaterals equilateral, leaving some vaguely apprehended general principle of continuity to do the rest. But the teacher should always insist on a detailed examination of the possibilities of the figure. Take the simplest instance: if the angle $APB=AQB$, then $APQB$ is cyclic only in the case when P, Q are on the same side of AB. Quite recently our Editor has elsewhere drawn attention to the fact that if the external bisectors of two angles of a triangle are equal, it does not necessarily follow that the triangle is isosceles.

In conclusion, we may make special mention of the elegance of the solutions of Nos. 31, p. 58; 67, p. 133; 109, p. 167; 59, p. 155. All solutions involving inversions are very satisfactorily done, and will repay careful study. We have also read with pleasure those relating to the contact of circles.

To pp. 111-119 attaches a somewhat melancholy interest. In the middle eighties keen enthusiasm was evoked by the New Geometry of the Triangle. Abroad Lemoine, Brocard, d'Ocagne, Neuberg, Vigarié, and at home Tucker, Taylor, Casey, and Simmons, vied with one another in developing new properties. Has all this interesting work failed to make any permanent mark on the Geometry of to-day? Perhaps the swing of the pendulum may again bring in something more interesting to geometricians than the factorisation of high integers. Let us hope that an attractive and informing work like this volume of solutions may prove the starting point for fresh geometrical development, by giving a stimulus to some dormant geometrical instinct.

R. F. DAVIS.

(1) **Premiers Principes de Géométrie Moderne.** By E. DUPORCQ. Pp. vii., 160. 3 fr. 1899. (Gauthier-Villars.)

(2) **The "Junior" Euclid, Books III. and IV.** By S. W. FINN. Pp. vi., 144–344. 2s. 1901. (Clarendon Press.)

(3) **Plane and Solid Geometry.** By A. SCHULTZE and F. L. SEVENOAK. Pp. xii., 370. 6s. 1901. (Macmillan.)

(4) **Elementary Geometry, Plane and Solid.** By THOMAS F. HOLGATE. Pp. xii., 440. 6s. 1901. (Macmillan.)

(5) **Famous Geometrical Theorems and Problems, with their History.** By W. W. RUPERT. Four Parts. Parts I. and II. Pp. 58. Ten cents each part. 1900. (Heath, Isbister.)

(6) **On Teaching Geometry.** By FLORENCE MILNER. Pp. 18. Ten cents. 1901. (Heath, Boston.)

(7) **Euclid I.–IV., VI., XI.** By C. SMITH and SOPHIE BRYANT. Pp. viii., 460. 4s. 6d. 1901. (Macmillan.)

(8) **Euclid I. and II.** By T. VARLEY. Pp. iv., 151. Second Edition. 1s. 1901. (Allman.)

(9) **Deductions in Euclid.** By T. W. EDMONDSON. Pp. 206. 2s. 6d. 1901. (Clive.)

(10) **Eléments de Géométrie.** Par F. J. Pp. xii., 523. 3 fr. 60 c. 1900. (Poussielgue.)

(11) **Leçons de Géométrie Elémentaire.** Vol. II. (Géométrie dans l'espace.) By J. HADAMARD. Pp. xxii., 582. 10 fr. 1901. (Armand Colin.)

(12) **Essai sur les fondements de la Géométrie.** By B. A. W. RUSSELL (translated by A. CADENAT). Pp. x., 274. 1901. (Gauthier-Villars.)

(13) **Les Principes Fondamentaux de la Géométrie.** By D. HILBERT (translated by L. LAUGEL). Pp. 114. 1900. (Gauthier-Villars.)

(14) **Traité de Géométrie Descriptive.** By J. DE LA GOURNERIE. In. 4. In three parts, each with Atlas, 1891, 1880, 1901. Editions, 3rd, 2nd, and 3rd respectively. 10 frs. each part. (Gauthier-Villars.)

(1) In his introduction the author laments the tendency of the ordinary official course to extend the study of analytical geometry at the expense of Geometry proper. "It seems to me that the study of the works of Poncelet, Chasles, and Laguerre would contribute more than an examination of the different methods of discussing the equation in S to develop the geometrical sense, which after all is the principal aim of the study of Mathematics." (We cannot follow M. Duporcq all the way in the last statement.) He has therefore set himself the task of making the geometric point of view as attractive as possible to the student, forced by the iron laws of a State "programme" to spend the greater part of his time in the uncongenial regions of analysis. This task he has accomplished with considerable success.

The wonderful fertility of modern geometry is due to the introduction of the idea of transformation, by means of which, from a single proposition, we can deduce a number of propositions distinct from the original. These transformations, moreover, may be successive. Double reciprocation is common enough. The Rev. T. C. Simmons developed the properties of harmonic polygons by reciprocations followed by orthogonal projections. Mr. C. E. M'Vicker has in these pages combined inversions with subsequent reflexions.

Poncelet's principle of continuity, extending the generality of algebraical language to geometry, serves the author as a *point de départ*. He then gives, in its most general form, the idea of transformation, throwing into bold relief the notion of contact transformation. The simplest transformations are those which transform points into points and lines into lines, and are respectively called *ponctuel* and homographic. After dealing with homographic divisions, pencils, etc., M. Duporcq proceeds to dwell on those transformations in which to a point there corresponds a line, and not another point. The simplest of these *dualistic* transformations is that in which to collinear points correspond concurrent lines. This doctrine of *correlation* naturally follows that of homographic transformation. Now, since homography leaves undisturbed the order of an algebraic curve or surface, and correlation transforms order into class, we see that any application of either to a conic or a quadric will give us properties connected with other conics or quadrics. The value, therefore, of these transformations in the theory of curves and surfaces is obvious. The last chapter deals with various other geometrical transformations, and in particular with inversions—"transformation par rayons vecteurs réciproques," and on those quadratic transformations of which inversion is only a particular case. The author then introduces us to Lie's transformations, in which lines are replaced by spheres. So much for the doctrine. As for the applications, the fourth and fifth chapters present in a really elegant and concise manner the principal properties of conics and quadrics: the conic, the range or pencil of conics, conics harmonically circumscribed or apolar to a conic, and so on. As a study in logical order, these chapters should make a marked impression on the mind of the student.

Those who have read Miss Scott's articles in recent numbers of the *Gazette* will hardly accept the following statement: "Le point . . . ne représente rien de géométrique, lorsqu'il est imaginaire, pas plus qu'une surface à coefficients imaginaires, le tétraèdre de référence étant supposé réel." This is enough to make von Staudt turn in his grave! There is a misprint of (80) for (86) in the note at the foot of p. 107. We notice that the theory of inversion is ascribed to Bellavitis (1836). Bellavitis was the first to make a general statement of the method, but the principle of inversion is traced as far back as Ptolemy by Chasles (*Rapport Hist.*, pp. 140-2), and according to Dr. C. Taylor, Quetelet was acquainted with it in 1827. Müller's German-French *Mathematisches Vokabularium* gives Cramer, 1750, and Magnus, 1832. Who was this Magnus? Can it be the Professor of Chemistry at Berlin (1802-1870)? No "Magnus" appears in the histories of Rouse Ball or Fink. But a "Magnus" wrote on *Die Abweichung der Geschosse* in 1852. If they are the same, he must have been a very Helmholtz for versatility.

(2) Mr. Finn's edition of Euc. III., IV. is on the lines of the corresponding edition of Books I., II. He draws special attention to his treatment of the identical equality of two triangles, which consists in having a separate picture of the triangles, the sides and angles bearing characteristic marks denoting the three equalities which enable the student to apply the results of Euc. I. 4, 8, 26.

In most of the figures further assistance is given to the student by drawing the lines of the data thicker than those of the construction. The Geometry of the Triangle is not neglected, and on p. 303 is a figure rarely to be found in our text-books—the nine point circle in contact with the in- and ex-circles. There is no mention of Wallace's claim to be considered the discoverer of the pedal line, the theorem in connection with which "is attributed to Robert Simpson." Inversion is just touched on in two pages, but poles and polars, radical axes, coaxal circles, etc., receive fuller treatment. The introduction to Book IV. is likely to prove useful, and the sections on Maxima and Minima and the doctrine of limits will be found adequate for the purposes of the pupil at this stage of his work. In addition to the various riders appended to the propositions there is a collection of easy and carefully graduated riders from Cambridge and Oxford local papers and other sources.

Mr. Finn has not hesitated, where experience has shown it desirable, to divide into parts with separate figures propositions such as III. 7, 8, 31, etc., and he has also included in the text the converse of III. 21, 22, 32, 35, 36. The young teacher will find much that is useful to him in the exercise of his craft in the pages of this edition of Euclid's Elements.

(3) In general plan and order of treatment the work of Messrs. Schultze and Sevenoak does not seem to differ essentially from other American Geometries. But it is especially remarkable for the deliberate attempt that is made by the authors to induce the student to think for himself. For instance, in the propositions the whole of the demonstration is not necessarily given, a hint as to the line of proof to be taken being often considered just enough to set the pupil on his road to the goal. Thus, in a very systematic manner the student is introduced into the mysteries of the "rider." Algebraical language is used without stint in dealing with proportion, and of course in the sections on Arithmetical Geometry. In the section on Regular Polygons the authors show that the circumference of a circle is less than the perimeter of any enveloping line, and that too by a method, the validity of which is not obvious. About 135 pages are devoted to solid and elementary spherical geometry. The diagrams in this division of the book are the best we have seen in any text-book. The prefatory hints to teachers contain several suggestions that may be found useful to those whose experience is small. The difficulty felt by beginners in remembering the data and the hypothesis is met by the use of coloured crayons and other graphic devices.

We have noticed a few slips, such as the following: In the list of symbols and abbreviations that for "is greater than" is wrong; and the small "s" inside $\angle$, $\triangle$, or $\square$ is trying to the eyes and more difficult to make when writing fast than if attached outside. A Scholium is defined as a remark, but §§ 81, 82 are headed respectively by the two terms as if a Remark were not quite the same thing as a Scholium. A circle is defined as a portion of a plane, etc., and we are asked to circumscribe a circle about a triangle. The solutions of the quadratic, p. 222, line 2, is incorrect; in the 2nd, 3rd, 5th, and 6th lines b and a should be interchanged, and the "construction" suffers accordingly.

(4) Professor Holgate covers exactly the same ground as Messrs. Schultze and Sevenoak, but treats the subject rather more fully. After the introductory chapter the pupil familiarises himself with the use of the ruler and compasses, and is thus "introduced to the idea of a formal proof in connection with matters which he clearly sees need proving." As in every day life the circle is never thought of as a portion of a plane; it is considered throughout the book as a particular kind of line, just as the polygon is a figure made up of points and lines. The aim of the authors of these books is to cultivate the geometric sense and the acquisition of geometric knowledge. To these aims logical reasoning and rhetorical demonstration are held as quite secondary objects. The exercises are in all cases well selected, and are intended to be worked through in the order in which they come, that is to say, they are an integral part of the text. Here in England it is difficult to say whether a course of Geometry, such as is here indicated, attains the result desired. It should be worth while to have an experimental class in some of our larger schools, but it should be in charge of a young enthusiast who is not enamoured of Euclidean methods. There is no doubt that a very much larger amount of geometric knowledge can be acquired

in a very much shorter time; and in the case of boys who leave school at an early age, the advantage of having worked through a book covering this ground in the time they usually take to "learn" four books of Euclid, is obvious. The essential question is, will the pupil have the geometric sense in a higher stage of development after such a course than if he had taken up Euclid, or will he assert with vicious warmth that Euclid or Schultze was a "beast who wrote books for the third and fourth forms"?

(5) We have received from Messrs. Isbister the first two parts of Mr. Rupert's monograph. They contain five proofs that the sum of the three angles of every plane triangle are together equal to two right angles, 26 proofs of Euc. I. 47, and a chapter on the value of π, half of which is taken up with extracts from De Morgan's Budget of Paradoxes.

(6) The aim of Miss Milner's well-meaning little pamphlet may be described as the apotheosis of the syllogism. If this (the syllogistic) "method be used in conventional and formal demonstration, it becomes of the greatest value in original work." With this bold statement we cannot entirely agree, and, in addition, we might observe that the method has the disadvantage of giving the average student a distaste for the subject in its elementary stages.

(7) We have looked forward with much interest to the completion of this School edition of Euclid, partly because the authors are distinguished members of what was once the A.I.G.T. and partly because each has undeniable claims on the respect and attention of mathematical teachers in this country. *Place aux dames!* Mrs. Bryant ranks with another member of our Association—Miss Dorothea Beale—as the most prominent of our head-mistresses, but she has made a reputation in more fields than one. She is a logician as well as a mathematician, a leader of no mean repute in ethics as well as in pedagogy. As for the Master of Sidney, is not his name a household word as the author of works on Arithmetic, Algebra, and Conics? From such collaboration we have therefore a right to expect something of the best, and we are not disappointed. How then is this edition distinguished from the multitudinous rivals in the market at the present moment? To Euclid's "order and manner" the authors rigorously adhere, but are animated by no spirit of slavish devotion to the doctrine of verbal inspiration, which they lament has not yet disappeared from some of our schools. And as for the pupil, it should be "considered distinctly meritorious to depart from the exact words of text-book or teacher, provided that a sound proof is given." With regard to the question of references, raised in vain in the *Gazette* by Mr. Siddons (No. 27, p. 58), the authors point out "that these references are given for the convenience of the learner, to show him where to look for some knowledge he may have forgotten. Examiners stand in no need of such help, and by no means wish to impose on the memory of the students the heavy and useless burden of learning how to give accurately numbered references. . . ." Remarks such as these serve to show the eminently sane manner in which the authors approach their subject.

In the First Book we find that no alternative proofs are given for Props. v. and vi., because "experience . . . appears to show that the average beginner finds Euclid's proofs easier to understand—or at any rate easier to reproduce—than the alternatives that have been suggested." We wonder if there is really any solid ground for this contention. It would seem impossible that anything could be simpler than the proof by superposition. As far as the experience of examiners goes, it may well be that the weaker vessels have been invariably taught what seemed to their teachers the easier method of proof, and that after the manner of their kind they distinguished themselves under the stress of the examination room. We note that the ordinary "proof" of Prop. xxiv. has disappeared, and that in Prop. xxii. it is shown that the circles must intersect. The common ambiguity in the use of the term "circle" is pointed out. Simson's enunciations of xviii. and xix. are rejected, useful notes are added to Props. xxxv., xliv., xlvi., xlvii. To Prop. xii. is added a proof that all right angles are equal; Prop. B. is a necessary addition to Prop. xxvi.; and "to describe a triangle equal to a given rectilineal figure" is given with Prop. xlv. from Thomas Simpson's Geometry. There is a good collection of worked-out theorems and problems, including a nice group of fifteen properties connected with squares described on the sides of *any* triangle. That the rectangle contained by the sum and difference of two straight

lines is equal to the difference of the squares on the lines is shown in the note to II. vi., and independently, after Props. ix. and x. To produce a given straight line so that the rectangle contained by the whole line thus produced and the part produced may be equal to a given square, is also given a place in Book II. Lardner's proofs of II. xii., xiii., as an extension of I. xlvii., are wisely given as alternatives. They are to be found in no other modern text-book, and the authors rightly claim that they show directly the equality of the figures and are more instructive than Euclid's proofs. There is a set of seventeen easy exercises on the figure of II. xi. The treatment of Book II. is most successful, but we think that it might have been left to the teacher to decide when a boy is to be permitted to write AB^2 for the sq. on AB, or $AB \cdot CD$ for the rect. AB, CD.

In Book III. the more notable changes are the proofs of Props. ix. and x., and the departure from the Euclidean method of treating the contact of circles. Props. xxvi.-xxix. are proved by superposition. Adequate attention is drawn to the relation between properties of tangents and properties of secants, and to the properties of intersecting and touching circles. In Book IV. we may notice the notes on Prop. x. as very instructive.

Under the title "Theory of Proportion" we have Euclid's test applied to six cases. Book VI. is followed by the usual collection of theorems in similitude, maxima and minima, inversion, etc. At the end of Book XI. are sections dealing with the more elementary properties of the sphere and tetrahedron. More examples might have been set on the "isosceles" tetrahedron, which lends itself very readily to investigation, and which we have found students delight in for their first and innocent experiments in original research. The authors have done well in making the pupil familiar with the names of geometers. There are a few lines on Euclid and his work, and we could wish that space had been found for more in the shape of biographical notes. We extend to the authors our hearty congratulations on the addition of a valuable and instructive text-book to the mathematical literature of our schools, an addition which we feel sure will attain a deservedly large measure of success.

(8) Mr. Varley's Euclid I. and II. has reached a second edition. Evidently some teachers find that beginners get on better when confronted with:—General enunciation, Specific ditto, (*a*) what is given, (*b*) what we want to do, or prove, how to do it or prove it. Where alternative proofs are given, the older proofs are in the text or in the appendix. Axiom xii. becomes Prop. 28A. The exercises and riders are carefully graduated, and a few additional propositions are worked out in full. A well-thought-out book for the beginner.

(9) Mr. Edmondson's book of deductions from the first four books of Euclid is evidently meant for the private student who takes such an examination as the London Matriculation. It is neatly and clearly printed, and most of the diagrams are very well drawn. That on p. 17 is incorrect, for CD is not the shortest side. The average reader will, we should think, find many of the demonstrations proved in even too much detail. But for the class for whom such a book is necessary this is erring on the right side.

(10) The *Frères des Écoles Chrétiennes* have well deserved their success as teachers, and French educational literature would be the poorer if the publications of this body were withdrawn from circulation. Of those publications the best are those devoted to Mathematics, and of these again, the best are the book under notice, and its complement, the corresponding *Livre du Maître*, entitled *Exercices de Géométrie*. The "Éléments" covers the ordinary ground of a French Geometry: Plane and Solid Geometry, and the "usual curves," *i.e.* the parabola, ellipse, hyperbola, and the helix. But the appendix of some two hundred pages takes the student much wider afield. In addition to poles and polars, inversion, directly and inversely similar figures, and the like, he is introduced to the modern geometry of the triangle, to surfaces of the second degree, to the helicoid, spirals, the cycloid and epicycloid, and the catenary. Guldin's theorems, areas and volumes, rules such as Simpson's rule, are discussed and applied to cases such as arise in the construction of bridges, tunnels, canals, etc. About a thousand theorems and problems are worked out in full, and as many more are set for solution. And all this for 3 fr. 50 c. The teacher who has the *Livre du Maître* will have therein the full solutions of over 2500 questions, of which nearly

300 are on the modern geometry of the triangle. As a mine of questions this collection will be difficult to beat. Both books are in thick boards, and will stand hard wear.

(11) This is the sequel to the *Leçons de Géométrie Plane* published in the course of elementary text-books edited by M. Darboux, the well-known Dean of the *Faculté des Sciences de Paris*, whose name needs no commendation to English mathematicians. And this volume is worthy of its place in a series which numbers among its contributors men like Messrs. Jules Tannery, Tisserand, and Andoyer. Although written expressly for students who propose to take the "baccalaureat," it does not merely confine itself to that programme, but may be taken as a complete exposition of elementary modern geometry. To this statement there is but one exception, the author does not deal with the conception of imaginaries. Among the novelties of treatment we notice a very pretty proof of Euler's theorem on polyhedra, in which, by the introduction of the idea of "order of connection," the author claims to have obtained the simplest existing proof, *du moins parmi les demonstrations correctes (on sait qu'il en existe beaucoup d'autres)*. He also ingeniously applies the theory of groups to the proof of the existence of regular polyhedra, introducing in particular "l'importante notion de domaine fondamentale." A remarkable note on the solubility of problems in Geometry is based on the researches of MM. Giacomini, Castelnuovo, and Enriques, recently published under the title *Questioni regnardanti la Geometria elementare*. Question 1020 runs as follows: "Consider the middle points C, C' of any two of the edges of a regular polyhedron. There is a plane regular polygon, having its centre at the centre of the circumsphere of the polyhedron, and its vertices at the middle points of certain of the edges, including C and C'. Then if this polygon have λ sides, and if m be the number of sides of each face of the polyhedron, n the number of edges of each polyhedral angle, c_λ, a_λ the side and in-radius of a λ-sided regular polygon, we have the following relations:

$$(c_\lambda/2)^2=(c_m/2)^2-a_n{}^2=(c_n/2)^2-a_m{}^2."$$

From this M. Hadamard obtains a new expression for the sides of the polyhedra inscribed in a given sphere:

$$2Rc_\lambda/c_n\,;\quad 2Rc_\lambda/c_m.$$

In some of the exercises the author introduces a *sorte particulière de géométrie*, in which some, but not all, of the relations between the elements of a triangle hold good. Other exercises give the elementary properties of quadrics, Dupin's cyclides, and the anallagmatic properties of the figure formed by two circles in space.

(12) We are glad to see a French translation of Mr. Russell's admirable essay on the Foundations of Geometry. The translation will, moreover, have a value of its own until we see a second edition of the English original, for it contains many corrections and additions, some on points of detail, and some in answer to the objections raised by critics. Unfortunately the reflections of the most eminent of these critics—M. Poincaré—reached the author too late to enable him to deal with them in this translation. Professor Halsted will be pleased to hear that Mr. Russell has abandoned his heresy *ré* the independence of Bolyaï and Gauss. M. Couturat, whose excursions into lands no less distant than the region of mathematical infinity will be familiar to some of our readers, has enriched the book with a lexicon of philosophical terms, which will be welcomed by the mathematical reader; and Mr. Russell has added mathematical notes in an appendix, for which the average philosophical reader will probably not be quite so grateful. He deserves cordial congratulation on the interest aroused in both the mathematical and the philosophical worlds by his essay—an interest which is acknowledged in fitting terms in his prefatory note to the labours of Professor Cadenat.

(13) M. Laugel has translated into French D. Hilbert's famous *Grundlagen de Geometrie*. The author has endeavoured to lay down complete and simple groups of mutually independent axioms from which he deduces important geometrical theorems in such a way as to exhibit effectually the *rôle* played by these axioms, and the conclusions which may be logically drawn therefrom. The groups are five in number, and are classed as axioms of associa-

tion, distribution, congruence, and continuity, with the "axiom of parallels." The only spatial axioms are Nos. 3-7 in group I., all the rest are either linear or "planar." They show us more than ever the truth of the saying of M. Poincaré: all axioms are but definitions in disguise. But in the present case the disguise is easily pierced. After demonstrating by means of these weapons a particular case of Pascal's Theorem, and the ordinary laws for the theory of plane areas, we come to Desargue's theorem on the intersection of the joins of the homologous vertices of triangles whose homologous sides are parallel. We are shown that it is impossible to prove this theorem without the aid of either the axioms of congruence or of spatial axioms. From a new segmentary calculus based on this theorem we are led to the equation of a straight line, and ultimately to the construction of a Geometry of space. In the German edition the author does not deal with the possibility of discussing a Geometry without the axiom of parallels, or with points as elements coupled with the idea of groups of displacements as in Sophus Lie's "travaux fondamentaux et féconds." But a few interesting remarks on the subject have been added to this translation, and of these Prof. Halsted has made use in his review of Manning's Non-Euclidean Geometry, in the *Gazette*, No. 29, p. 94.

(14) The handsome volumes of M. Gournerie on Descriptive Geometry take us back to the days of Chasles and Poncelet. The first edition appeared in three parts from 1860 to 1864. Poncelet alludes to it in his *Traité des propriétés progressives des figures* as the most complete, the most accurate, and "le plus rationnel" of any work on this subject that had as yet appeared. Chasles became its sponsor in a very practical way, by speaking highly of it before the *Académie des Sciences*. The third edition of the third part has been edited by Professor Lebon, of the Lycée Charlemagne, who succeeded the author in his chair in the *Conservatoire des Arts et des Métiers*, and who is, moreover, the author of a large work on the same subject. Among the more interesting features of the part which has just appeared, we may mention the simple and elegant presentation of the theory of Curvature of Surfaces. In this, as well as in the proof of Euler's formula giving the curvature of a normal section, the infinitesimal calculus is not used, the author founding his treatment on Bertrand's Theorem (Salmon, *Geom. of Three Dim.*, p. 265, note). We find Meunier's Theorem (*loc. cit.*, p. 256) applied to the construction of the radii of curvature and the osculating planes of a curve given by its projections. The author follows Dupin's treatment of the lines of curvature of surfaces of the second order. M. Lebon brings the book up to date by notes historical and illustrative.

Differential and Integral Calculus. By E. W. Nichols. Pp. xii., 394. 7s. 6d. 1900. (Heath, Boston.)

This is essentially a book for beginners, by which we mean that the author has laboured to present his subject in the clearest and simplest manner, removing "all obscurities and mysteries," and smoothing the path of the student generally. Accordingly there is more explanatory matter than is generally to be found in a book for "the undergraduate courses of our best Universities, colleges, and technical schools." In some ways a book of this type is very useful to the tyro, especially if he be a private student. But there is always the danger that in removing every statement that may cause the student to exercise his wits, we are losing an opportunity of stimulating his attention and cultivating a valuable faculty. Mr. Nichols is to be commended for the skilful way in which the geometrical, mechanical, and electrical applications are worked in throughout the book. The historical notes are concise and to the point. Some twenty pages are given to differential equations. The volume is well bound and printed, and the author's boast that he gives the reader a "clear and open" page is amply justified.

Differential and Integral Calculus for Beginners. By E. Edser. Pp. vi. 253. 2s. 6d. 1901. (Nelson.)

Yet another Calculus "adapted to the use of Students of Physics and Mechanics," "shorn of all extraneous difficulties," providing the "physical student with a valuable engine of research," the student finding "no difficulties which cannot be overcome by application and perseverance"! Assuming on

the part of the reader only an elementary knowledge of Algebra and Geometry, the author proceeds at once to initiate him into the mysteries of co-ordinate geometry and of circular and potential functions. The proofs of trigonometrical formulae required are relegated to an appendix. The differentiation of simple and complex functions occupy pp. 27-53; then follow two chapters on maxima and minima and expansions of functions. Two chapters on integration lead up to applications to geometrical, mechanical, and physical problems. Double and triple integrations and easy differential equations bring the book to a close. The treatment is throughout clear and simple; in fact, many of the problems are worked out with almost unnecessary detail of development. For instance, the moment of inertia of an ellipsoid drags through $2\frac{1}{2}$ pages, and requires neither "application" nor "perseverance" to follow its course. But, no doubt, this is quite necessary with students whose mathematical equipment is limited in character. We notice a curious mistake on page 101. Here the author brings the mean ordinate to $y = a \sin bx$ into relation with the maximum of alternating electric currents. He seems to think that "the maximum current is a little greater than $1\frac{1}{2}$ times the average current." A reference to Everett (Deschanel, part iii., p. 176) will show that this cannot be true. Has the author confused square root of mean square into mathematical mean?

The Elements of Hydrostatics. By S. L. Loney, M.A. Pp. x. 248. 4s. 6d. (Cambridge University Press.)

The peculiar characteristics of Mr. Loney's works are so familiar to most teachers that it is unnecessary to dwell on them here. Suffice it to say that the *Hydrostatics* is no exception to the rule. The chapter on Centres of Pressure strikes us as more complete than is usually the case in an elementary work. The valves in the Condenser (p. 180) might be more convincing. Why not have substituted for this picture a simple machine in daily use such as the bicycle pump? (cf. Greenhill's *Hydrostatics*, p. 374). The sections dealing with curves of buoyancy and tensions of vessels are as simple as is necessary for ordinary students. We think that some definite reasons should be given why the theory of Hydrostatics here laid down should be even remotely applicable to other than the "perfect liquid," and that, in general, more illustrative matter is advisable referring to the machines in daily use. It is really remarkable how differently a student works at theories which have obvious practical applications. We *must* be wrong if we neglect any method likely to convince the pupil of the practical value and relevance of his investigations.

Eléments de Mathématiques Supérieures à l'usage des Physiciens, Chimistes et Ingénieurs, et des élèves des Facultés des Sciences. By H. Vogt. Pp. vii., 619. 10 fr. 1901. (Nony, Paris.)

Nancy is one of the few Universities in Europe where the study of Physics and Chemistry is so well organised that students can avail themselves of the services of a Professor of Mathematics to unveil those mysteries of his science which are indispensable to their course of instruction. This book contains about fifty chapters, each about the length of a lecture, beginning with equations containing two unknowns, and ending with linear differential equations. From easy determinants, the binomial theorem, surds and indices, the author proceeds to the discussion of the doctrine of series, prefacing his remarks on series by generalities on limits. The usual rules on convergence, etc., are followed by a detailed account of method of approximation to the sum of a series. Logarithms and the exponential theorem complete the section devoted to Algebra, with the exception of the few notes on elimination and series at the end of the book. Analytical Geometry is introduced by general ideas on units, etc., the examples being derived from Geometry, Mechanics, and Physics. Then come co-ordinates and the representation of lines and surfaces by means of equations. Elementary Differential Calculus with its geometrical applications leads up to the Integral, which is carried as far as the theorems of Ostrogradsky, Stokes, and Green. The book closes with some two hundred exercises on all the sections. This rapid summary will give an idea of the ground covered by these lectures, and as they have been "professed" by the author for several years, they no doubt represent the minimum that is required by those who wish to have at their disposal in the shortest possible period a mathematical armoury for prac-

tical application. The explanatory matter is full without being verbose, and the style is simple and clear. This volume will not detract from the reputation of Professor Vogt.

PROBLEMS.

[Owing to pressure of space, these must be discontinued until the solutions of Nos. 1-449 are completed.]

SOLUTIONS.

The following problems, published in Vol. I. are still unsolved: 171, 275, 279, 283, 326, 336-8, 341, 349, 370, 373-4, 376-9, 381-2, 389, 395-7. Solutions to these, or to problems in Vol. II., should be sent as early as possible.

The question need not be re-written; the number should precede the solution. **Figures** should be **very carefully drawn to a small scale** on a separate sheet.

356. [I. 17. a.] *Find three consecutive numbers each of the form* $\square+\square$. (*Ex.* $2248=42^2+22^2$; $2249=43^2+20^2$; $2250=45^2+15^2$.) *I have six solutions with numbers under* 1000, *and eight with numbers between* 1000 *and* 2000. *Can anyone indicate more solutions between these limits?* W. ALLEN WHITWORTH.

Solution by R. F. DAVIS.

There are seven solutions with numbers under 1000; namely, 72-4, 232-4, 288-90, 520-2, 584-6, 800-2, 808-10. Also there are ten solutions with numbers between 1000 and 2000; namely, 1096-8, 1152-4, 1224-6, 1312-4, 1600-2, 1664-6, 1744-6, 1800-2, 1872-4, 1960-2. Of these three, viz. 288-290, 1224-6, and 1600-2, include numbers (289, 1225, 1600) which although expressible as sums of squares are themselves square numbers: these sets, therefore, Mr. Whitworth rejected.

Particular formulae are

$$\left.\begin{aligned}\{a(a+1)\}^2+\{a(a+1)\}^2&=2a^4+4a^3+2a^2\\ \{a(a+2)\}^2+(a^2-1)^2&=2a^4+4a^3+2a^2+1\\ \{a(a+1)+1\}^2+\{a(a+1)-1\}^2&=2a^4+4a^3+2a^2+2\end{aligned}\right\},$$

which gives four of the above solutions, including one of the rejected; and

$$\left.\begin{aligned}(2a^3-a^2-a)^2+(2a^3+a^2+a)^2&=8a^6+2a^4+4a^3+2a^2\\ (2a^3-a^2)^2+(2a^3+a^2+1)^2&=8a^6+2a^4+4a^3+2a^2+1\\ (2a^3-a^2+a+1)^2+(2a^3+a^2-a+1)^2&=8a^6+2a^4+4a^3+2a^2+2\end{aligned}\right\},$$

which gives two more of the solutions.

Solution by C. E. YOUNGMAN.

The sum of two squares is either $4x$ (both even), $4x+1$ (even and odd), or $8x+2$ (both odd); so that three consecutive sums must have the forms $8n$, $8n+1$, $8n+2$. And one of these must be divisible by 3; $\therefore$ by 9, for a square is either $9y$ or $3y+1$;

$$\therefore\quad n=9m,\ 9m+1,\text{ or }9m+2. \qquad \text{.....(A)}$$

Also $8n=(2a)^2+(2b)^2$; $\therefore$ a and b are both odd or both even, and

$$4n=(a-b)^2+(a+b)^2=4c^2+4d^2;$$

$\therefore$ n is either a square (when $a=b$) or the sum of two unequal squares;...(B)

$$\therefore\quad n=4p,\ 4p+1,\text{ or }8p+2. \qquad \text{.....(C)}$$

(A) and (C) combined show that n must have one of the forms: $36r$, $36r+1$, $72r+2$, $72r+10$, $36r+9$, $72r+18$, $36r+20$, $36r+28$, $36r+29$.

Now for solutions below 1000, $n<125$; proceed as follows:

Write down the possible values of n; strike out all that do not satisfy (B). [To be the sum of two unequal squares a number must be a prime of the form $4x+1$, or a product of such primes, multiplied may be by powers of 2 and squares.]

n				$4n+1$				$8n+1$			
	36	72*	108*		145				289		
1	37	73	109	5	149	293	437*	9*	297*	585	
2*		74				297*					
9	45	81	117	37	181	325	469*	73	361*	649*	
10		82		21*		329*					
18*		90				361				721*	
20	56*	92*		81				161*			
28*	64	100			257	401			513*	801	
29	65	101		117	261	405		233	521	809	

The numbers marked * being thus struck out, find the values of $4n+1$ for those that remain; if $8n+2$ be the sum of two squares, $4n+1$ is either a square or the sum of two squares; strike out as before those values which do not satisfy this condition; and, finally, write down the values of $8n+1$ for the numbers left and strike out those which are not of the form $\square+\square$, and we are left with the middle numbers of seven solutions; and in the same way between 1000 and 2000 may be found ten more solutions, viz.: $8n+1=1097, 1153, 1225, 1313, 1601, 1665, 1745, 1801, 1873, 1961$.

369. [K. 5. d.] *If triangles ABC, $A'B'C'$ be in perspective (centre P, axis DEF), and be also orthologic—i.e. such that perpendiculars from the vertices of either on corresponding sides of the other concur (in Q and Q')—then PQQ' is a straight line perpendicular to DEF.*

Cor. 1. *If PQ be points in the plane of a triangle ABC such that perpendiculars from Q on AP, BP, CP, cut BC, CA, AB on a straight line, then so also do perpendiculars from P on AQ, BQ, CQ; and both the lines are perpendicular to PQ.*

Cor. 2. *If $ABCD$ be four points determining a rectangular hyperbola, the tangent at D may be had thus:—Draw DE perpendicular to BD cutting CA at E, and DF perpendicular to CD cutting AB at F; then DT perpendicular to EF is the tangent.* C. E. Youngman.

Solution by C. E. M'Vicker.

Consider P the vertex of a triangular pyramid standing on ABC. Then DEF is the line common to the planes ABC, $A'B'C'$. If now the plane $A'B_1C_1$ be drawn through A' parallel to ABC, its line of intersection $A'D_1$ with $A'B'C'$ will be parallel to DEF. The axis of perspective of $A'B'C'$ and $A'B_1C_1$ is therefore parallel to DEF. It is then only necessary to prove the theorem for the orthologic triangles $A'B'C'$, $A'B_1C_1$ having the vertex A' in common.

Draw B_1I, C_1J perpendicular to $A'D_1$ meeting QB', QC' in S and T, the latter meeting $A'B_1$, $A'C_1$ in U and V. UV is clearly perpendicular to QA' and therefore parallel to BC, B_1C_1.

$$B_1S/B_1U=\sin SUB_1/\sin USB_1,$$

$$C_1T/C_1V=\sin TVC_1/\sin VTC_1,$$

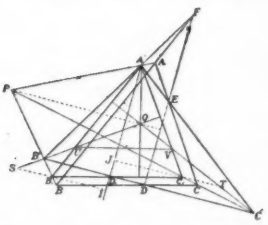
and $$B_1U/B_1A'=C_1V/C_1A' \text{ by parallels};$$

$$\therefore \frac{B_1S}{C_1T}=\frac{A'B_1\sin SUB_1\sin VTC_1}{A'C_1\sin TVC_1\sin USB_1}=\frac{A'B_1\sin IA'B_1}{A'C_1\sin IA'C_1};$$

$$[\because \sin SUB_1=\cos A=\sin TVC_1]$$

$$\therefore B_1S/C_1T=B_1I/C_1J, \text{ so that } S \text{ and } T \text{ lie in directum with } D_1.$$

The sides of the triangles $PB'C'$, ∞ST therefore pass through the collinear points $B_1D_1C_1$, their base angles lying on QB', QC'; $\therefore$ the line ∞P joining their vertices passes through Q. Hence PQ is parallel to SI, TJ and therefore at right angles to DEF.



Otherwise: let $\alpha\beta\gamma$ and xa, $y\beta$, $z\gamma$ be vectors from the centre of perspective P to the vertices of ABC, $A'B'C'$, and let the vector $PQ\equiv\rho$. It is easily seen that

$$\sigma\equiv x(y-z)\alpha+y(z-x)\beta+z(x-y)\gamma$$

is a vector parallel to the axis of perspective of the triangles.

Since $\rho-\alpha$ is perpendicular to $y\beta-z\gamma$ we have, by quaternions,

$$S(\rho-\alpha)(y\beta-z\gamma)=0.$$

Similarly, $$S(\rho-\beta)(z\gamma-x\alpha)=0,$$

and $$S(\rho-\gamma)(x\alpha-y\beta)=0.$$

Multiply in order by the scalars x, y, z and add, so as to eliminate terms independent of ρ; then we obtain $S\rho\sigma=0$, proving that ρ is at right angles to σ.

Mr. M'Vicker adds: I have not succeeded in deducing the corollaries from the proposition itself, but give in outline an analytical proof of a property suggested by them and including both.

$ABCDE$ are five coplanar points. Perpendiculars through E to DA, DB, DC meet BC, CA, AB in collinear points PQR; it is required to prove that $ABCDE$ lie on a rectangular hyperbola, and that PQR is perpendicular to DE. Conversely, if they lie on a rectangular hyperbola, the perpendiculars from E to DA, DB, DC meet BC, CA, AB in collinear points.

Take as coordinates of the five points $(t_1, 1/t_1)$, $(t_2, 1/t_2)$, $(t_3, 1/t_3)$, $(t_4, 1/t_4)$, $(t_5, 1/\theta)$, the axes being the asymptotes of the rectangular hyperbola $xy=1$ through the first four.

Form the equations of EP, EQ, ER; P, Q, R being the intersections of BC. CA. AB with the line $ax+by=1$: express that these lines are perpendicular to DA, DB, DC, and eliminate a and b; we then find a determinant equation in θ which reduces to

$$\begin{vmatrix} t_1\theta+t_2t_3, & t_1t_5+t_2t_3, & 1 \\ t_2\theta+t_3t_1, & t_2t_5+t_3t_1, & 1 \\ t_3\theta+t_1t_2, & t_3t_5+t_1t_2, & 1 \end{vmatrix}=0.$$

This can only be satisfied by $\theta=t_5$ which proves that the fifth point lies on the hyperbola.

The converse is evident on putting $\theta = t_5$ at the beginning. If now we find $a:b$ from the equations which gave the determinant we get the ratio $t_4 t_5 : -1$, which shows that PQR is perpendicular to DE.

This latter property is also evident *a priori* since EP, EQ, ER are the sides of an evanescent triangle in orthologic perspective with ABC.

387. [K. 17. e. 20. f.] *Construct a spherical quadrilateral given* α, β, γ, δ, *the mid points of the sides taken in order; and prove that the cosine of half its spherical excess is equal to*

$$\cos\beta\gamma\cos\alpha\delta - \cos\gamma\alpha\cos\beta\delta + \cos\alpha\beta\cos\gamma\delta.$$ C. E. M'VICKER.

Solution and Note by Proposer.

If $ABCD$ be the spherical quadrilateral, a, b, c, d the sides, DA, AB, BC, CD, and $E \equiv A+B+C+D-2\pi$ the spherical excess, then, by a known property of a triangle, $\alpha\beta$ meets DB at a point ω distant $\frac{\pi}{2}$ from O the middle point of DB; hence also $\gamma\delta$ meets DB at the same point ω. Also if ωX and ωP be taken along $\omega\beta$, ωB equal to $\alpha\beta$, OB respectively; then the triangle ωPX is right-angled at P, and PX is half the spherical excess of ABD. M'Clelland and Preston's *Spherical Trigonometry*, § 104.

Produce $\alpha\beta . \gamma\delta$ to meet in ω; and on $\omega\beta . \omega\gamma$ take $\omega X . \omega Y$ equal respectively to $\beta\alpha . \gamma\delta$; draw the great circle $\omega PBOD$ perpendicular to XY; mark off ωO a quadrant and BO, OD each equal to ωP; then B, D are two vertices of the required quadrilateral, and XY is half its spherical excess.

Again $\cos\frac{1}{2}E = \cos XY = \cos\omega X . \cos\omega Y + \sin\omega X . \sin\omega Y . \cos\Omega$;

$$\therefore \cos\tfrac{1}{2}E = \cos\alpha\beta . \cos\gamma\delta + \sin\alpha\beta . \sin\gamma\delta . \cos\Omega,$$

where Ω denotes the angle $X\omega Y$.

Take now the trigonometrical identity

$$\begin{aligned}\sin(\lambda-\lambda')\sin(\mu-\mu')\cos\Omega \\ \equiv (\cos\lambda\cos\mu+\sin\lambda\sin\mu\cos\Omega)(\cos\lambda'\cos\mu'+\sin\lambda'\sin\mu'\cos\Omega) \\ -(\cos\lambda\cos\mu'+\sin\lambda\sin\mu'\cos\Omega)(\cos\lambda'\cos\mu+\sin\lambda'\sin\mu\cos\Omega),\end{aligned}$$

and make $\lambda = \text{arc}\,\omega\alpha$, $\lambda' = \omega\beta$, $\mu = \omega\delta$, $\mu' = \omega\gamma$;

then immediately we obtain

$$\sin\alpha\beta . \sin\gamma\delta . \cos\Omega = \cos\alpha\delta . \cos\beta\gamma - \cos\gamma\alpha . \cos\beta\delta,$$

whence $$\cos\tfrac{1}{2}E = \cos\beta\gamma . \cos\alpha\delta - \cos\gamma\alpha . \cos\beta\delta + \cos\alpha\beta . \cos\gamma\delta,$$

as was to be proved.

From this simple formula, E can be readily expressed in terms of the sides and diagonals of $ABCD$.

For by an elementary theorem,

$$\cos\alpha A + \cos\alpha B = 2\cos\beta A . \cos\alpha\beta.$$

Multiply each by $2\cos\alpha A$ and denote DB, AC by e, f, then

$$1+\cos a+\cos b+\cos e = 4\cos\frac{a}{2}.\cos\frac{b}{2}+\cos\alpha\beta;$$

$$\therefore\ 2\cos\frac{a}{2}\cos\frac{b}{2}.\cos\alpha\beta = \cos^2\frac{a}{2}+\cos^2\frac{b}{2}+\cos^2\frac{e}{2}-1.$$

Similarly $$2\cos\frac{c}{2}\cos\frac{d}{2}.\cos\gamma\delta = \cos^2\frac{c}{2}+\cos^2\frac{d}{2}+\cos^2\frac{e}{2}-1,$$

„ $$2\cos\frac{b}{2}\cos\frac{c}{2}.\cos\beta\gamma = \cos^2\frac{b}{2}+\cos^2\frac{c}{2}+\cos^2\frac{f}{2}-1,$$

„ $$2\cos\frac{a}{2}\cos\frac{d}{2}.\cos\alpha\delta = \cos^2\frac{a}{2}+\cos^2\frac{d}{2}+\cos^2\frac{f}{2}-1,$$

similarly $$2\cos\frac{a}{2}\cos\frac{c}{2}.\cos\gamma\alpha=\cos^2\frac{e}{2}+\cos^2\frac{f}{2}+\cos^2\frac{b}{2}+\cos^2\frac{d}{2}-2,$$

„ $$2\cos\frac{b}{2}\cos\frac{d}{2}.\cos\beta\delta=\cos^2\frac{e}{2}+\cos^2\frac{f}{2}+\cos^2\frac{a}{2}+\cos^2\frac{c}{2}-2.$$

But by the preceding theorem,

$$\cos\tfrac{1}{2}E=\cos\beta\gamma.\cos\alpha\delta-\cos\gamma\alpha.\cos\beta\delta+\cos\alpha\beta.\cos\gamma\delta,$$

hence after reduction we find

$$4\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}\cos\frac{d}{2}.\sin^2\frac{1}{4}E$$
$$=\sin^2\frac{e}{2}\sin^2\frac{f}{2}-\left(\cos\frac{a}{2}\cos\frac{c}{2}-\cos\frac{b}{2}\cos\frac{d}{2}\right)^2,$$

the formula given by Serret, *Méthodes de Géométrie*; also Casey, § 119; M'Clelland and Preston, § 186.

As far as I am aware this latter result has not hitherto been obtained without the aid of stereographic projection, nor has the theorem in this article been given.

388. [K. 11. e.] *The line ABCD is cut harmonically in B and C; on AC and BD as diameters circles APC, BQD are described whose planes intersect at right angles, and PQ, PB, QC are drawn. Shew that*

$$PQ.BC=PB.QC.$$ A. S. Toller.

Solution by W. F. Beard.

Bisect AC, BD at E, F. Draw a $\odot$ on AC as diameter in the plane of the paper.

From F draw $FG\perp$r. to AD meeting PC produced in G.

With centre G, radius GB, describe a sphere cutting PC in H, K.

$GC.GP$ = sq. of tangent from G to $\odot APC$
$=GE^2-EC^2$
$=FG^2+EF^2-EC^2$ .....I. 47.
$=FG^2+FB^2$ for $EF^2=EC^2+FB^2$ see (α)
$=GB^2$ .....I. 47.
$=GH^2$;

$\therefore$ $PH.CK$ is a harmonic range.

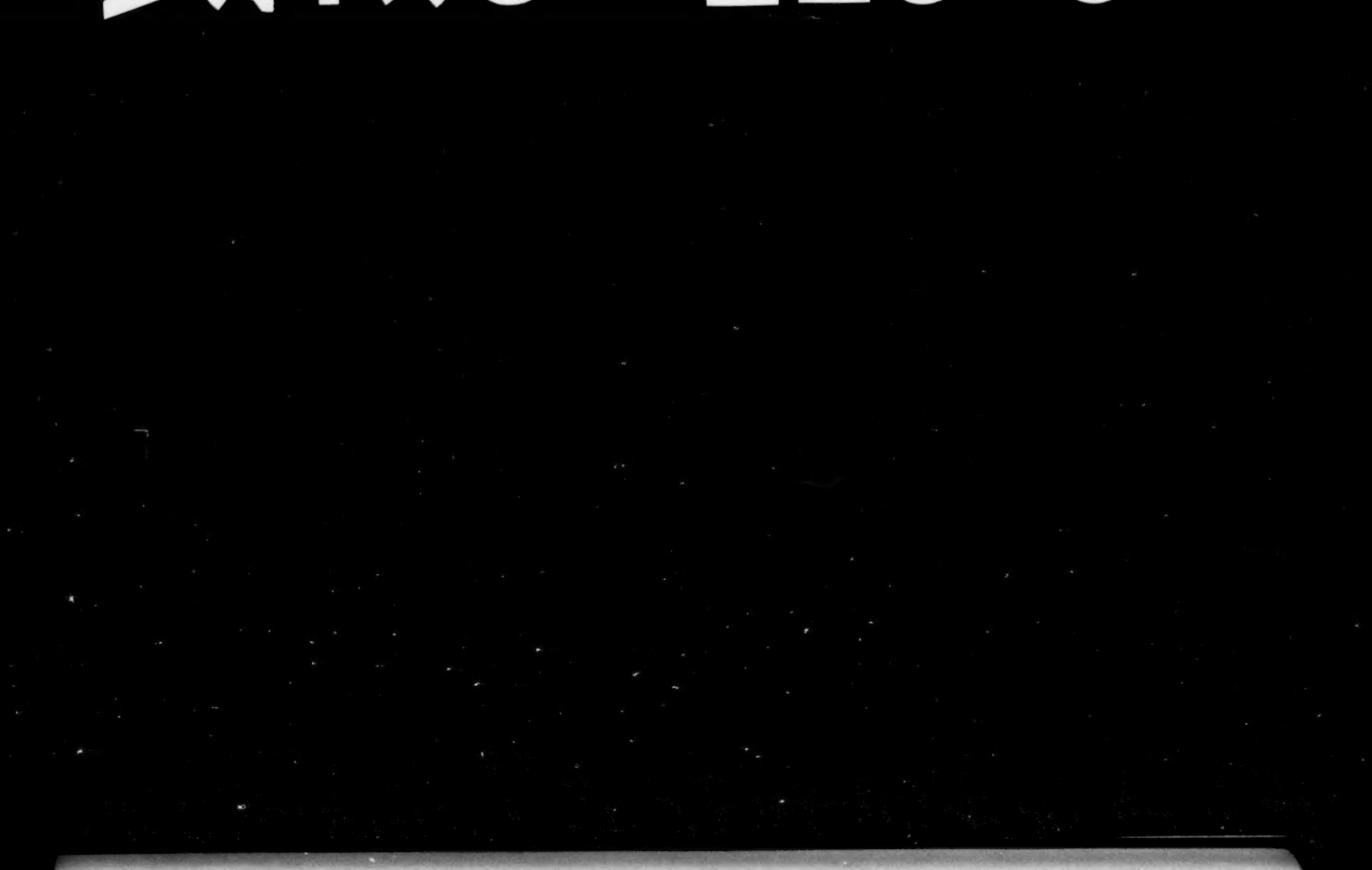
Also since $GB^2=FB^2+FG^2$; $\therefore$ $\odot$ on BD as diameter at right angles to the plane of the paper is a section of the sphere, centre G, radius GB.

Thus Q lies on this sphere;

$\because$ $PHCK$ is a harmonic range;
$\therefore$ $PQ:QC=PH:HC$ from (β), (see below)
and $PB:BC=PH:HC$ „ „
$\therefore$ $PQ:QC=PB:BC$
or $PQ.BC=PB.QC.$ Q.E.D.

(α) If $ABCD$ is a harmonic range, the $\odot$s. on AC, BD as diameters cut orthogonally.

(β) If $ABCD$ is a harmonic range, and P any point on the $\odot$ on AC as diameter, then $PB:PD=BC:CD=AB:AD$.

Slide Rule.—Professor Barrell informs us, with reference to his paper (Vol. II., No. 29) on the Slide Rule, that Messrs. A. G. Thornton, of Manchester, have already prepared a rule with the index line on the cursor 'broken,' as he suggested. The rule also gives an extra $\frac{3}{16}$" of 'dead end' at each end of the scale.

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